

# ON THE TORAL RANK CONJECTURE FOR GRADED 3-STEP NILPOTENT LIE ALGEBRAS

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**ABSTRACT.** An algebraic version of the Toral Rank Conjecture states that  $\dim H^*(\mathfrak{n}) \geq 2^{\dim \mathfrak{z}}$  for any finite dimensional nilpotent Lie algebra  $\mathfrak{n}$  with center  $\mathfrak{z}$ . If  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \dots \oplus \mathfrak{n}_k$  is graded nilpotent, then Deninger and Singhof proved that  $\dim H^*(\mathfrak{n}) \geq L(p)$  where  $p(x) = (1-x)^{d_1} \dots (1-x^k)^{d_k}$ ,  $d_i = \dim \mathfrak{n}_i$  and  $L(p)$  is the sum of the absolute values of the coefficients of  $p$ . It follows from this result that the TRC holds for 2-step nilpotent Lie algebras. A natural question is whether it is also possible to derive the TRC for 3-step nilpotent Lie algebras. In this paper we construct a family of graded 3-step nilpotent Lie algebras  $\mathfrak{n}(n)$ ,  $n \in \mathbb{N}$ , such that, if  $n \geq 17$ , then  $L(p) < 2^{\dim \mathfrak{z}}$  for all possible gradings of the form  $\mathfrak{n}(n) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ . For  $n = 17$ , we have  $\dim \mathfrak{n}(n) = 212$ ,  $\dim \mathfrak{n}_1 = 20$ ,  $\dim \mathfrak{n}_2 = 156$ ,  $\dim \mathfrak{n}_3 = 36$ ,  $\dim \mathfrak{z} = 189$  and  $\frac{L(p)}{2^{\dim \mathfrak{z}}} \approx 0.906$ . On the other hand,  $\mathfrak{n}(17)$  admits a grading  $\mathfrak{n}(17) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathfrak{n}_6 \oplus \mathfrak{n}_7 \oplus \mathfrak{n}_8 \oplus \mathfrak{n}_{12}$  such that  $L(p) > 2^{\dim \mathfrak{z}}$ .

We also show computationally that  $L(p) > 2^{\dim \mathfrak{z}}$  for any indecomposable graded 3-step nilpotent Lie algebra  $\mathfrak{n}(n) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$ , with  $\dim \mathfrak{n} < 99$ , and therefore the TRC holds for 3-step graded nilpotent Lie algebras  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$  of dimension less than 99.

## 1. INTRODUCTION AND RESULTS

The Toral Rank Conjecture (TRC) was formulated by Halperin [H] more than 25 years ago and an algebraic version of it is the following:

**TRC.** If  $\mathfrak{n}$  is a finite dimensional nilpotent Lie algebra with center  $\mathfrak{z}(\mathfrak{n})$ , then

$$\dim H^*(\mathfrak{n}) \geq 2^{\dim \mathfrak{z}(\mathfrak{n})}.$$

This conjecture has a topological origin: the toral rank  $r(X)$  of a differentiable manifold  $X$  is the dimension of the greatest torus acting freely on  $X$ . Originally, the TRC states that the cohomology of the manifold  $X$  has dimension greater than or equal to  $2^{r(X)}$ . It follows from a theorem of Nomizu [N] that, for compact nilmanifolds, the original TRC would follow from the algebraic version stated above.

This conjecture remains open in general and it has only been proved for certain classes of nilpotent Lie algebras, for example when  $\mathfrak{n}$  is 2-step nilpotent (see [DS]), or split metabelian (see [PT]).

The TRC for 2-step nilpotent Lie algebras is a consequence of a bound for  $\dim H^*(\mathfrak{n})$  obtained by Deninger and Singhof [DS] for graded nilpotent Lie algebras over a field of characteristic zero. This result is stated below as Theorem 1.2, but first we need the following definition.

**Definition 1.1** ([DS]). *Given a graded nilpotent Lie algebra (always assumed to be finite dimensional)  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \dots \oplus \mathfrak{n}_k$ , the polynomial associated to the grading is*

$$p(x) = (1-x)^{d_1} (1-x^2)^{d_2} \dots (1-x^k)^{d_k}, \quad \text{where } d_i = \dim(\mathfrak{n}_i).$$

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Recall that given a polynomial  $p(x) = \sum a_i x^i$ , its length  $L(p)$  is

$$L(p) = \sum |a_i|.$$

**Theorem 1.2** ([DS]). *If  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_k$  is a graded nilpotent Lie algebra over a field of characteristic zero and  $p$  is the associated polynomial, then*

- (a)  $\dim H^*(\mathfrak{n}) \geq L(p)$ ,
- (b)  $L(p) \geq 2^D$  with  $D = \max\{d_i : \frac{k}{2} < i \leq k\}$ , and
- (c)  $L(p) \geq q^{\dim \mathfrak{n}}$  with  $q = p^{\frac{1}{p-1}}$ ,  $p$  a prime greater than  $k$ .

**Definition 1.3.** *Let  $\mathfrak{n}$  be a graded nilpotent Lie algebra. We will say that:*

- $\mathfrak{n}$  is long-polynomial type (LPT) if it admits a grading such that

$$(1.1) \quad L(p) \geq 2^{\dim \mathfrak{z}(\mathfrak{n})}.$$

- $\mathfrak{n}$  is weakly short-polynomial type (WSPT) if  $\mathfrak{n}$  is  $k$ -step nilpotent and the inequality (1.1) does not hold for any grading  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \cdots \oplus \mathfrak{n}_k$ .
- $\mathfrak{n}$  is short-polynomial type (SPT) if the inequality (1.1) does not hold for any grading of  $\mathfrak{n}$ .

Theorem 1.2 implies that the TRC holds for any LPT Lie algebra. As Deninger and Singhof point out, it is straightforward to obtain from item (b) that every 2-step nilpotent Lie algebra is LPT. It is also clear that Theorem 1.2 implies the TRC for any many special classes of graded nilpotent Lie algebras, for instance those  $\mathfrak{n}$  for which its center is contained in  $\mathfrak{n}_k$ , such as the nilradicals of parabolic subalgebras.

Now, it is natural to ask whether it is possible to obtain from this theorem the TRC for graded 3-step nilpotent Lie algebras and in particular if these Lie algebras are LPT. On the other hand, SPT Lie algebras are potential counterexamples to the TRC. Motivated by these questions, in §2 we verify computationally that if  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$  is an indecomposable graded 3-step nilpotent Lie algebra with  $\dim(\mathfrak{n}) < 99$ , then  $\mathfrak{n}$  is LPT. Using Künneth Formula, it is immediate that every graded 3-step nilpotent Lie algebra  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$  of dimension less than 99 satisfies the TRC.

On the other hand, in §3 we construct a family of graded 3-step nilpotent Lie algebras  $\mathfrak{n}(n)$ ,  $n \in \mathbb{N}$ , such that  $\mathfrak{n}(n)$  is WSPT for  $n \geq 17$ . We verified computationally that many members of this family are LPT. The original motivation of this family was to construct potential counterexamples to the TRC, but these examples show that WSPT does not imply SPT. More precisely, we prove that for any grading  $\mathfrak{n}(n) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$  the dimensions of the subspaces  $\mathfrak{n}_i$ , for  $i = 1, 2, 3$ , are fixed. These dimensions are:

$$\dim(\mathfrak{n}_1) = n + 3, \quad \dim(\mathfrak{n}_2) = 3 + \frac{n(n+1)}{2}, \quad \dim(\mathfrak{n}_3) = 2(n+1).$$

This is achieved by carefully studying the Lie algebra of derivations of  $\mathfrak{n}(n)$ . We obtain, in particular, the Levi decomposition of  $\text{Der}(\mathfrak{n}(n))$  (see Theorem 3.7).

For these Lie algebras we have

$$\dim(\mathfrak{n}(n)) = 5 + \frac{(n+1)(n+6)}{2}, \quad \dim(\mathfrak{z}) = \frac{(n+1)(n+4)}{2},$$

and for  $n = 17$  (the first  $n$  for which  $\mathfrak{n}(n)$  is WSPT), we have

$$\dim(\mathfrak{n}(n)) = 212, \quad \dim \mathfrak{z} = 189, \quad \text{and} \quad \frac{L(p)}{2^{\dim \mathfrak{z}}} \approx 0.906.$$

On the other hand,  $\mathfrak{n}(17)$  admits a grading  $\mathfrak{n}(17) = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3 \oplus \mathfrak{n}_6 \oplus \mathfrak{n}_7 \oplus \mathfrak{n}_8 \oplus \mathfrak{n}_{12}$  such that the inequality (1.1) holds.

2. TRC FOR GRADED 3-STEP NILPOTENT  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$  WITH  $\dim \mathfrak{n} < 99$ 

If  $\mathfrak{n}_1$  and  $\mathfrak{n}_2$  are ideals of a Lie algebra  $\mathfrak{n}$  such that  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ , then it follows from Künneth formula (see for instance [W]) that  $H^k(\mathfrak{n}) = \sum_{i+j=k} H^i(\mathfrak{n}_1) \otimes H^j(\mathfrak{n}_2)$ , for all  $k$  and, in particular,

$$(2.1) \quad H^*(\mathfrak{n}) = H^*(\mathfrak{n}_1) \otimes H^*(\mathfrak{n}_2).$$

Recall that a Lie algebra is indecomposable if it cannot be written as the sum of two non trivial ideals. Since  $\mathfrak{z}(\mathfrak{n}_1 \oplus \mathfrak{n}_2) = \mathfrak{z}(\mathfrak{n}_1) \oplus \mathfrak{z}(\mathfrak{n}_2)$  it follows from (2.1) that it is enough to prove the TRC for indecomposable Lie algebras.

Let  $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2 \oplus \mathfrak{n}_3$  (from now on, the direct sums considered are as vector spaces) be an indecomposable graded 3-step nilpotent Lie algebra. By graded 3-step nilpotent we will mean that it is not 2-step nilpotent and in particular  $\mathfrak{n}_i \neq 0$  for  $i = 1, 2, 3$ . It also follows that

$$(2.2) \quad \mathfrak{z} = \mathfrak{n}_3 \oplus \mathfrak{z} \cap [\mathfrak{n}_1, \mathfrak{n}_1], \quad \text{and} \quad [\mathfrak{n}_1, \mathfrak{n}_2] = \mathfrak{n}_3.$$

Let  $\mathfrak{z}_2 = \mathfrak{n}_2 \cap \mathfrak{z} \subset [\mathfrak{n}_1, \mathfrak{n}_1]$ , let  $\mathfrak{n}_2^1$  be a complementary subspace of  $\mathfrak{z}_2$  in  $[\mathfrak{n}_1, \mathfrak{n}_1]$  and let  $\mathfrak{n}_2^0$  be a complementary subspace of  $[\mathfrak{n}_1, \mathfrak{n}_1]$  in  $\mathfrak{n}_2$ . That is,

$$(2.3) \quad [\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2^1 \oplus \mathfrak{z}_2, \quad \text{and} \quad \mathfrak{n}_2 = \mathfrak{n}_2^0 \oplus [\mathfrak{n}_1, \mathfrak{n}_1] = \mathfrak{n}_2^0 \oplus \mathfrak{n}_2^1 \oplus \mathfrak{z}_2.$$

If  $d_2^0 = \dim(\mathfrak{n}_2^0)$ ,  $d_2^1 = \dim(\mathfrak{n}_2^1)$ ,  $z_2 = \dim(\mathfrak{z}_2)$  and  $z = \dim(\mathfrak{z})$ , then it is clear that

$$d_2 = d_2^0 + d_2^1 + z_2 \quad \text{and} \quad z = d_3 + z_2.$$

Moreover, we have the following inequalities.

**Proposition 2.1.** *The numbers  $d_1$ ,  $d_2^0$ ,  $d_2^1$ ,  $z_2$  and  $d_3$  satisfy:*

- (a)  $d_1, d_2^1, d_3 > 0$ .
- (b)  $d_1(d_2^0 + d_2^1) \geq d_3$ .
- (c)  $\binom{d_1}{2} \geq d_2^1 + z_2$ .
- (d)  $d_1 d_3 \geq d_2^0$ .

*Proof.* Since  $\mathfrak{n}$  is not 2-step nilpotent, we have (a). Since  $\mathfrak{n}$  is indecomposable, it follows from the above definitions that the linear map

$$\mathfrak{n}_1 \otimes (\mathfrak{n}_2^0 \oplus \mathfrak{n}_2^1) \rightarrow \mathfrak{n}_3$$

induced by the bracket is surjective, and thus we obtain (b). Similarly, (c) follows from (2.3) and the surjectivity of the linear map  $\Lambda^2 \mathfrak{n}_1 \rightarrow [\mathfrak{n}_1, \mathfrak{n}_1]$  induced by the bracket. Finally, since  $\mathfrak{z} \cap \mathfrak{n}_2^0 = \{0\}$ , it follows that

$$\text{ad} : \mathfrak{n}_2^0 \rightarrow \text{Hom}(\mathfrak{n}_1, \mathfrak{n}_3)$$

is injective, and therefore we obtain (d).  $\square$

We verified computationally there are 5-tuples  $(d_1, d_2^0, d_2^1, z_2, d_3)$  of non-negative integers satisfying (a), (b), (c) and (d) of the previous proposition and

$$L((1-x)^{d_1}(1-x^2)^{d_2}(1-x^3)^{d_3}) < 2^{z_2+d_3},$$

(with  $d_2 = d_2^0 + d_2^1 + z_2$ ) such that  $d_1 + d_2^0 + d_2^1 + z_2 + d_3 < 99$ . For each 5-tuple  $(d_1, d_2^0, d_2^1, z_2, d_3)$  we set

$$F = \frac{L((1-x)^{d_1}(1-x^2)^{d_2}(1-x^3)^{d_3})}{2^{z_2+d_3}}, \quad \text{with } d_2 = d_2^0 + d_2^1 + z_2.$$

The following table shows, for some  $d$ 's (most of them odd between 5 and 106), the minimum value of  $F$  among all 5-tuples  $(d_1, d_2^0, d_2^1, z_2, d_3)$  of non-negative integers satisfying  $d = d_1 + d_2^0 + d_2^1 + z_2 + d_3$  and (a), (b), (c) and (d) of Proposition 2.1.

$d$	$d_1$	$d_2^0$	$d_2^1$	$z_2$	$d_3$	$F$	$d$	$d_1$	$d_2^0$	$d_2^1$	$z_2$	$d_3$	$F$
5	2	1	1	0	1	8.00	61	9	0	2	32	18	2.42
7	3	0	1	1	2	4.50	63	9	0	2	34	18	2.30
9	3	0	1	2	3	3.44	65	10	0	2	33	20	2.34
11	4	0	1	2	4	3.81	67	10	0	2	35	20	2.22
13	4	0	1	4	4	3.62	69	10	0	2	37	20	2.12
15	5	0	1	4	5	4.02	71	10	0	2	39	20	2.03
17	5	0	1	6	5	3.88	73	10	0	2	41	20	1.95
19	5	0	1	8	5	3.78	75	10	0	2	43	20	1.87
21	6	0	1	8	6	4.20	77	10	0	3	34	30	1.91
23	6	0	1	10	6	4.08	79	10	0	3	36	30	1.77
25	6	0	1	12	6	3.98	81	10	0	3	38	30	1.64
27	6	0	1	14	6	3.89	83	10	0	3	40	30	1.53
29	7	0	1	14	7	4.31	85	10	0	3	42	30	1.42
31	6	0	2	11	12	4.07	89	11	0	3	42	33	1.37
33	6	0	2	13	12	3.81	87	10	2	1	44	30	1.33
35	7	0	1	20	7	4.04	91	11	0	3	44	33	1.28
37	7	0	2	14	14	3.84	93	11	0	3	46	33	1.19
39	7	0	2	16	14	3.59	95	11	0	3	48	33	1.11
41	7	0	2	18	14	3.37	97	11	0	3	50	33	1.04
43	7	1	1	20	14	3.20	98	11	0	3	51	33	1.01
45	8	0	2	19	16	3.36	99	11	0	3	52	33	0.97
47	8	0	2	21	16	3.16	100	11	1	2	53	33	0.94
49	8	0	2	23	16	2.98	101	11	2	1	54	33	0.91
51	8	0	2	25	16	2.81	102	12	0	3	51	36	0.96
53	8	1	1	27	16	2.65	103	12	0	3	52	36	0.92
55	9	0	2	26	18	2.81	104	12	0	3	53	36	0.90
57	9	0	2	28	18	2.67	105	12	0	3	54	36	0.87
59	9	0	2	30	18	2.54	106	12	0	3	55	36	0.84

### 3. EXISTENCE WSPT 3-STEP NILPOTENT LIE ALGEBRAS

In this section we construct a family  $\mathfrak{n}(n)$ ,  $n \in \mathbb{N}$ , of graded 3-step nilpotent Lie algebras such that  $\mathfrak{n}(n)$  is WSPT for  $n \geq 17$ .

**3.1. Definition of the family  $\mathfrak{n}(n)$ .** In what follows, if  $A$  is a set,  $\langle A \rangle$  will denote the free vector space with  $A$  as a basis. For each positive integer  $n$ , let

$$E_n = \langle \{e_i : i = 1, \dots, n\} \rangle, \quad U_n = \langle \{u_i : i = 1, \dots, n\} \rangle, \\ X_n = \langle \{x_i : i = 1, \dots, n\} \rangle, \quad Y_n = \langle \{y_i : i = 1, \dots, n\} \rangle.$$

Since  $n$  will be fixed most of the time, we will use  $E$ ,  $U$ ,  $X$  and  $Y$  to denote the spaces  $E_n$ ,  $U_n$ ,  $X_n$  e  $Y_n$ . We will define on the vector space

$$\mathfrak{n} = \mathfrak{n}(n) = E \oplus \langle \{a, b, x, u, y\} \rangle \oplus \Lambda^2 E \oplus \langle \{c\} \rangle \oplus X \oplus U \oplus Y \oplus \langle \{f, h\} \rangle$$

a Lie algebra structure that makes it graded 3-step nilpotent. We start by choosing the subspaces  $\mathfrak{n}_1$ ,  $\mathfrak{n}_2$  and  $\mathfrak{n}_3$  corresponding to the grading of  $\mathfrak{n}$ .

Let

$$\mathfrak{n}_1 = E \oplus \langle \{a, b, x\} \rangle, \\ \mathfrak{n}_2 = \langle \{u, y\} \rangle \oplus \Lambda^2 E \oplus \langle \{c\} \rangle \oplus X, \\ \mathfrak{n}_3 = U \oplus Y \oplus \langle \{f, h\} \rangle,$$

and let

$$\mathfrak{B}_1 = \{e_1, e_2, \dots, e_n, a, b, x\}, \\ \mathfrak{B}_2 = \{u, y\} \cup \{e_i \wedge e_j : 1 \leq i < j \leq n\} \cup \{c\} \cup \{x_i : 1 \leq i \leq n\}, \\ \mathfrak{B}_3 = \{u_1, \dots, u_n, y_1, \dots, y_n, f, h\}.$$

be ordered bases of  $\mathfrak{n}_1$ ,  $\mathfrak{n}_2$  and  $\mathfrak{n}_3$  respectively (we choose the lexicographic order for  $e_i \wedge e_j$ ). Now

$$\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3$$

is an ordered basis of  $\mathfrak{n}$ . It is clear that

$$d_1 = \dim(\mathfrak{n}_1) = n + 3, \\ d_2 = \dim(\mathfrak{n}_2) = \frac{n(n+1)}{2} + 3, \\ d_3 = \dim(\mathfrak{n}_3) = 2(n+1).$$

We now define the Lie bracket of  $\mathfrak{n}$  in terms of this basis as shown in the following table:

$[\mathfrak{n}, \mathfrak{n}]$	$e_1$	$\dots$	$e_n$	$a$	$b$	$x$	$u$	$y$	$\Lambda^2 E$	$c$	$X$	$U$	$Y$	$f$	$h$
$e_1$	0	$\dots$	$e_1 \wedge e_n$	0	0	$x_1$	$u_1$	$y_1$	0	0	0	0	0	0	0
$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$e_n$	$e_n \wedge e_1$	$\dots$	0	0	0	$x_n$	$u_n$	$y_n$	0	0	0	0	0	0	0
$a$	0	$\dots$	0	0	$c$	0	0	$f$	0	$h$	0	0	0	0	0
$b$	0	$\dots$	0	$-c$	0	0	$h$	$h$	0	0	0	0	0	0	0
$x$	$-x_1$	$\dots$	$-x_n$	0	0	0	$f$	$h$	0	0	0	0	0	0	0
$u$	$-u_1$	$\dots$	$-u_n$	0	$-h$	$-f$	0	0	0	0	0	0	0	0	0
$y$	$-y_1$	$\dots$	$-y_n$	$-f$	$-h$	$-h$	0	0	0	0	0	0	0	0	0
$\Lambda^2 E$	0	$\dots$	0	0	0	0	0	0	0	0	0	0	0	0	0
$c$	0	$\dots$	0	$-h$	0	0	0	0	0	0	0	0	0	0	0
$X$	0	$\dots$	0	0	0	0	0	0	0	0	0	0	0	0	0
$U$	0	$\dots$	0	0	0	0	0	0	0	0	0	0	0	0	0
$Y$	0	$\dots$	0	0	0	0	0	0	0	0	0	0	0	0	0
$f$	0	$\dots$	0	0	0	0	0	0	0	0	0	0	0	0	0
$h$	0	$\dots$	0	0	0	0	0	0	0	0	0	0	0	0	0

**Remark 3.1.** Suppose that we change the basis  $B = \{e_i\}$  of  $E$  by  $B' = \{e'_i\}$ , and we define accordingly  $x'_i = [e'_i, x]$ ,  $y'_i = [e'_i, y]$ ,  $u'_i = [e'_i, u]$ . If we consider the following new ordered bases of  $\mathfrak{n}_1$ ,  $\mathfrak{n}_2$  and  $\mathfrak{n}_3$

$$\mathfrak{B}'_1 = \{e'_1, e'_2, \dots, e'_n, a, b, x\}$$

$$\mathfrak{B}'_2 = \{u, y\} \cup \{e'_i \wedge e'_j : 1 \leq i < j \leq n\} \cup \{c\} \cup \{x'_i : 1 \leq i \leq n\}$$

$$\mathfrak{B}'_3 = \{u'_1, \dots, u'_n, y'_1, \dots, y'_n, f, h\}$$

then the above bracket-table looks the same for the new basis  $\mathfrak{B}' = \mathfrak{B}'_1 \cup \mathfrak{B}'_2 \cup \mathfrak{B}'_3$  of  $\mathfrak{n}$ . From now on, we will think of  $\mathfrak{B}$  as a map that assigns a basis  $B'$  of  $E$  to  $\mathfrak{B}(B') = \mathfrak{B}'$ .

**Proposition 3.2.** For every  $n \in \mathbb{N}$ ,  $\mathfrak{n}(n)$  is a graded 3-step nilpotent Lie algebra with

$$d_1 = n + 3, \quad d_2 = \frac{n(n+1)}{2} + 3, \quad d_3 = 2n + 2,$$

$$d_2^0 = 2, \quad d_2^1 = 1, \quad z_2 = \frac{n(n+1)}{2}, \quad z = \frac{(n+4)(n+1)}{2}.$$

*Proof.* The only basis elements  $t, v, w$  such that  $[[t, v], w] \neq 0$  are  $t = b$  and  $v = w = a$ . Therefore, the Jacobi's identity is trivially satisfied in  $\mathfrak{n}$ .  $\square$

**3.2. Derivations of  $\mathfrak{n}(n)$ .** Let  $\text{Der}(\mathfrak{n})$  be the Lie algebra of derivations of  $\mathfrak{n}$ . In this subsection we will describe some properties of the matrices corresponding to elements in  $\text{Der}(\mathfrak{n})$  associated to a basis  $\mathfrak{B}(B)$ .

**Definition 3.3.** We denote by  $\text{Der}(\mathfrak{n})_0$  the subalgebra of derivations  $D$  such that

- $D(E) \subset E$  and
- $D(a) = D(b) = D(x) = D(u) = D(y) = 0$ .

It is clear that there is Lie algebra isomorphism

$$\mathfrak{gl}(E) \rightarrow \text{Der}(\mathfrak{n})_0$$

$$A \mapsto D_A.$$

The matrix  $[D_A]_{\mathfrak{B}(B)}$  of  $D_A$  in a basis  $\mathfrak{B}(B)$  is block-diagonal, where the blocks corresponding to each subspace are described by the following table:

$$(3.1) \quad \begin{array}{c|cccc|cccc|cccc} & E & a & b & x & u & y & \Lambda^2 E & c & X & U & Y & f & h \\ \hline [A]_B & 0 & 0 & 0 & 0 & 0 & 0 & [\Lambda^2 A]_{\Lambda^2 B} & 0 & [A]_B & [A]_B & [A]_B & 0 & 0 \end{array}$$

Here,  $\Lambda^2 B = \{e_i \wedge e_j : 1 \leq i < j \leq n\}$  and  $\Lambda^2 A$  is the linear map on  $\Lambda^2 E$  defined by  $\Lambda^2 A(e_i \wedge e_j) = A(e_i) \wedge e_j + e_i \wedge A(e_j)$ .

**Definition 3.4.** We denote by  $\text{Der}(\mathfrak{n})_1$  the set of derivations  $D$  such that  $D(E) \subset W$ , where  $W = \langle \{a, b, x, u, y\} \rangle \oplus \Lambda^2 E \oplus \langle \{c\} \rangle \oplus X \oplus U \oplus Y \oplus \langle \{f, h\} \rangle$ .

**Proposition 3.5.** If  $B$  is a basis of  $E$  and  $D \in \text{Der}(\mathfrak{n})_1$ , then the matrix of  $D$  in the basis  $\mathfrak{B}(B)$  is lower triangular.

*Proof.* We need to check that for every element  $w \in \mathfrak{B}(B)$ , the coordinates of  $Dw$  are zero on the basis vectors located left to  $w$ , according to the order in  $\mathfrak{B}(B)$ . We will use the following notation: if  $v \in \mathfrak{n}$  and  $w \in \mathfrak{B}(B)$ ,  $\lambda_w(v)$  will be the  $w$  coordinate of  $v$ .

Since  $D$  is a derivation, we know that

$$D(\mathfrak{n}') \subset \mathfrak{n}', \quad D([\mathfrak{n}, \mathfrak{n}']) \subset [\mathfrak{n}, \mathfrak{n}'], \quad D(\mathfrak{z}(\mathfrak{n})) \subset \mathfrak{z}(\mathfrak{n}).$$

In what follows, we will omit the parenthesis and write  $Dv$  for  $D(v)$ .

(1) From the definition of  $\mathfrak{n}$ , it is clear that  $[\mathfrak{n}, \mathfrak{n}'] = \langle h \rangle$ , so  $Dh = \lambda_h(Dh)h$ .

(2) Since  $c = [a, b]$  we have

$$\begin{aligned} Dc &= [Da, b] + [a, Db] \\ &= \lambda_a(Da)c - (\lambda_u(Da) + \lambda_y(Da))h + \lambda_b(Db)c + \lambda_y(Db)f + \lambda_c(Db)h \\ &= (\lambda_a(Da) + \lambda_b(Db))c + \lambda_y(Db)f + (-\lambda_u(Da) - \lambda_y(Da) + \lambda_c(Db))h. \end{aligned}$$

This is what we need for  $Dc$ .

(3) Since  $f = [x, u] = [a, y]$ , then  $[Dx, u] + [x, Du] = [Da, y] + [a, Dy]$ . In addition,

$$\begin{aligned} [Dx, u] &= \sum \lambda_{e_i}(Dx)u_i + \lambda_b(Dx)h + \lambda_x(Dx)f \\ [x, Du] &= -\sum \lambda_{e_i}(Du)x_i + \lambda_y(Du)h + \lambda_u(Du)f \\ [Da, y] &= \sum \lambda_{e_i}(Da)y_i + (\lambda_b(Da) + \lambda_x(Da))h + \lambda_a(Da)f \\ [a, Dy] &= \lambda_b(Dy)c + \lambda_c(Dy)h + \lambda_y(Dy)f \end{aligned}$$

and therefore

$$(3.2) \quad \lambda_b(Dy) = 0 \quad \text{and} \quad \lambda_{e_i}(Dx) = \lambda_{e_i}(Du) = \lambda_{e_i}(Da) = 0 \quad \text{for all } 1 \leq i \leq n.$$

We also conclude that

$$\begin{aligned} Df &= (\lambda_b(Dx) + \lambda_y(Du))h + (\lambda_x(Dx) + \lambda_u(Du))f \\ &= (\lambda_b(Da) + \lambda_x(Da) + \lambda_c(Dy))h + (\lambda_a(Da) + \lambda_y(Dy))f. \end{aligned}$$

This is what we need for  $Df$ .

(4) Since  $D \in \text{Der}(\mathfrak{n})_1$ , we know that  $D(e_i) \in W$  and hence, for all  $1 \leq i < j \leq n$ ,

$$\begin{aligned} D(e_i \wedge e_j) &= [D(e_i), e_j] + [e_i, D(e_j)] \\ &= -\lambda_x(De_i)x_j - \lambda_u(De_i)u_j - \lambda_y(De_i)y_j \\ &\quad + \lambda_x(De_j)x_i + \lambda_u(De_j)u_i + \lambda_y(De_j)y_i, \end{aligned}$$

and this is what we need to prove for  $D(e_i \wedge e_j)$ .

(5) Let  $1 \leq i \leq n$ . Since  $[a, e_i] = 0$ , then  $[Da, e_i] + [a, D(e_i)] = 0$  and

$$\begin{aligned} 0 &= \sum_j \lambda_{e_j}(Da)e_j \wedge e_i - \lambda_x(Da)x_i - \lambda_y(Da)y_i - \lambda_u(Da)u_i \\ &\quad + \lambda_b(De_i)c + \lambda_y(De_i)f + \lambda_c(De_i)h, \end{aligned}$$

therefore

$$\begin{aligned} 0 &= \lambda_{e_j}(Da) \text{ for all } 1 \leq j \leq n, \\ 0 &= \lambda_x(Da) = \lambda_y(Da) = \lambda_u(Da), \\ 0 &= \lambda_b(De_i) = \lambda_y(De_i) = \lambda_c(De_i), \text{ for all } 1 \leq i \leq n. \end{aligned}$$

Hence we proved, among other things, what is needed for  $Da$ .

(6) Since  $[b, e_i] = 0$ , then  $[Db, e_i] + [b, D(e_i)] = 0$  and taking into account that  $\lambda_y(De_i) = 0$  (see (5)) we obtain

$$0 = \sum_j \lambda_{e_j}(Db)e_j \wedge e_i - \lambda_x(Db)x_i - \lambda_u(Db)u_i - \lambda_y(Db)y_i - \lambda_a(De_i)c + \lambda_u(De_i)h$$

and hence

$$\begin{aligned} 0 &= \lambda_{e_j}(Db) \text{ for all } 1 \leq j \leq n, \\ 0 &= \lambda_x(Db) = \lambda_y(Db) = \lambda_u(Db), \\ 0 &= \lambda_a(De_i) = \lambda_u(De_i), \text{ for all } 1 \leq i \leq n. \end{aligned}$$

This is almost what we needed for  $Db$ . We now combine this and results from (2) and (5) to obtain

$$Dc = [Da, b] + [a, Db] = (\lambda_a(Da) + \lambda_b(Db))c + \lambda_c(Db)h,$$

and hence  $[Dc, b] = 0$ . Since  $[b, c] = 0$ , it follows that  $[Db, c] = 0$  and thus  $\lambda_a(Db) = 0$ . This completes what we need for  $Db$ .

(7) We now consider the cases of  $x, u, y$ .

(i) We first check that the  $a$  and  $b$  coordinates of  $Dx, Du, Dy$  are zero.

We start with  $Dx$ : since  $[b, x] = 0$ , we have  $[Dx, b] + [x, Db] = 0$ , and since  $\lambda_c([x, Db]) = 0$ , we obtain  $\lambda_c([Dx, b]) = 0$ . This implies that  $\lambda_a(Dx) = 0$ .

Repeating this argument and observing that  $[a, x] = 0$ , we will get that  $Dx$  has no  $b$  coordinate.

If now we do it considering  $[a, u] = 0$ , we will get that  $Du$  has no  $b$  coordinate.

We notice that the same argument, always analyzing the  $c$  coordinate, can be repeated using  $[u, b] = h$  and  $[b, y] = h$  respectively to conclude that  $Du$  and  $Dy$  don't have  $a$  coordinates.

We have already seen in (3.2) that  $Dy$  doesn't have a  $b$  coordinate.

(ii) Let us consider now the  $x$  and  $u$  coordinates of  $Dy$ :  $[u, y] = 0$  implies  $[Du, y] + [u, Dy] = 0$  and, since we know that the  $a$  coordinate of  $Du$  is 0, the  $f$  coordinate of  $[Du, y]$  is 0, and then  $[u, Dy]$  has no  $f$  coordinate, which implies that the  $x$  coordinate of  $Dy$  is 0.

Again, the same argument considering  $[x, y] = h$  leads us to conclude that the  $u$  coordinate of  $Dy$  is 0.

(iii) We consider now the  $x$  coordinate of  $Du$ . Being  $[u, y] = 0$ ,  $[Du, y] + [u, Dy] = 0$ . Looking at the  $h$  coordinate of this sum, we get

$$0 = \lambda_b(Du) + \lambda_x(Du) - \lambda_b(Dy).$$

We have just seen that  $\lambda_b(Dy) = 0 = \lambda_b(Du)$ , then  $\lambda_x(Du) = 0$ , as we need.

- (iv) Finally, we just need to prove that the  $u$  and  $e_i$  coordinates of  $Dy$  are zero. Since  $[x, y] = h$ , and recalling that  $\lambda_u(Dx) = 0$  and  $\lambda_{e_i}(Dx) = 0$ , we have

$$\begin{aligned}\lambda_h(Dh)h &= D(h) = [Dx, y] + [x, Dy] \\ &= (\lambda_x(Dx)h - \sum \lambda_{e_i}(Dy)x_i + \lambda_u(Dy)f + \lambda_y(Dy)h),\end{aligned}$$

so  $\lambda_u(Dy) = 0$  and  $\lambda_{e_i}(Dy) = 0$  for all  $1 \leq i \leq n$ .

- (8) For any  $1 \leq i \leq n$ , we have that

$$\begin{aligned}D(x_i) &= D([x, e_i]) = [Dx, e_i] + [x, D(e_i)] \\ &= -\lambda_x(Dx)x_i - \lambda_u(Dx)u_i - \lambda_y(Dy)y_i + \lambda_u(Dx)f + \lambda_y(Dx)h.\end{aligned}$$

On the other hand, since we know from (7.iii) that  $\lambda_x(Du) = 0$ ,

$$\begin{aligned}D(u_i) &= D([u, e_i]) = [Du, e_i] + [u, D(e_i)] \\ &= -\lambda_u(Du)u_i - \lambda_y(Du)y_i - \lambda_x(De_i)f - \lambda_b(De_i)h.\end{aligned}$$

Finally, having in mind that the  $x$  and  $u$  coordinates of  $Dy$  are 0 (see (7.ii)),

$$\begin{aligned}D(y_i) &= D([y, e_i]) = [Dy, e_i] + [y, D(e_i)] \\ &= -\lambda_y(Dy)y_i - \lambda_a(De_i)f - (\lambda_b(De_i) + \lambda_x(De_i))h.\end{aligned}$$

With this we conclude the cases of  $x_i$ ,  $u_i$  and  $y_i$  and the proof is complete.  $\square$

**Proposition 3.6.** *Let  $D \in \text{Der}(\mathfrak{n})_1$ . For each element  $v \in \mathfrak{B}(B)$ , we denote by  $\lambda_v$  the diagonal coefficient of the matrix of  $D$  corresponding to the vector  $v$ . Then:*

- (a)  $\lambda_{e_i} = 0$ ,  $1 \leq i \leq n$ .
- (b)  $\lambda_a = \lambda_b = \lambda_x$ .
- (c)  $\lambda_h = \lambda_f = 3\lambda_a$ .
- (d)  $\lambda_y = \lambda_u = \lambda_c = 2\lambda_a$ .

*Proof.* (a) is obvious from the definition. We will prove next (b), (c) and (d). From the previous proposition we know that the matrix of  $D$  is lower triangular. Hence:

- (1)  $[x, u] = f$  implies  $[D(x), u] + [x, D(u)] = D(f)$  and thus  $\lambda_x + \lambda_u = \lambda_f$ .
- (2)  $[a, y] = f$  implies  $[D(a), y] + [a, D(y)] = D(f)$  and thus  $\lambda_a + \lambda_y = \lambda_f$ .
- (3)  $[a, c] = h$  implies  $[D(a), c] + [a, D(c)] = D(h)$  and thus  $\lambda_a + \lambda_c = \lambda_h$ .
- (4)  $[b, y] = h$  implies  $[D(b), y] + [b, D(y)] = D(h)$ . We know from (6) in the proof of the previous proposition, that the  $u$  coordinate of  $D(b)$  is 0, and thus  $\lambda_b + \lambda_y = \lambda_h$ .
- (5)  $[b, u] = h$  implies  $[D(b), u] + [b, D(u)] = D(h)$ . Also from (6) in the proof of the previous proposition, we know that the  $y$  coordinate of  $D(b)$  is 0, and thus  $\lambda_b + \lambda_u = \lambda_h$ .
- (6)  $[x, y] = h$  implies  $[D(x), y] + [x, D(y)] = D(h)$ , and thus  $\lambda_x + \lambda_y = \lambda_h$ .
- (7)  $[a, b] = c$  implies  $[D(a), b] + [a, D(b)] = D(c)$ , and thus  $\lambda_a + \lambda_b = \lambda_c$ .

From (3) and (7), we have  $2\lambda_a + \lambda_b = \lambda_h$  and combining with (5), we obtain

$$(3.3) \quad 2\lambda_a = \lambda_u.$$

Substituting in (1),  $\lambda_x + 2\lambda_a = \lambda_f$ .

From (4) and (5), we have

$$(3.4) \quad \lambda_y = \lambda_u,$$

from (1) and (6), we have

$$(3.5) \quad \lambda_h = \lambda_f,$$

from (2), (4) and (6),  $\lambda_a = \lambda_b = \lambda_x$ , and this proves (b). From this and (3.5) we obtain (c), that is,  $\lambda_h = \lambda_f = 3\lambda_a$ .



From (3) and (4) it follows  $\lambda_y = \lambda_c$  and from this and (3.4) and (3.3), we obtain (d). This ends the proof of the proposition.  $\square$

As a consequence, we obtain the following theorem that describes the Levi decomposition of  $\text{Der}(\mathfrak{n})$ .

**Theorem 3.7.** *Let  $\text{Der}(\mathfrak{n})$  be the Lie algebra of derivations of  $\mathfrak{n}$ , and let  $\text{Der}(\mathfrak{n})_0$  and  $\text{Der}(\mathfrak{n})_1$  be the Lie subalgebras of  $\text{Der}(\mathfrak{n})$  defined previously. Then:*

- (a)  $\text{Der}(\mathfrak{n})_0$  is a Lie subalgebra of  $\text{Der}(\mathfrak{n})$  isomorphic to  $\mathfrak{gl}(E)$ .
- (b)  $\text{Der}(\mathfrak{n})_1$  is a solvable ideal of  $\text{Der}(\mathfrak{n})$ .
- (c)  $\text{Der}(\mathfrak{n}) = \text{Der}(\mathfrak{n})_0 \oplus \text{Der}(\mathfrak{n})_1$ .

*Proof.* (a) has been already discussed when we defined  $\text{Der}(\mathfrak{n})_0$ , and (b) is a consequence of the fact that the matrix of any  $D \in \text{Der}(\mathfrak{n})_1$  is lower triangular in any basis  $\mathfrak{B}(B)$  of  $\mathfrak{n}$ .

To prove (c), let us see first that the sum is direct. If  $D \in \text{Der}(\mathfrak{n})_0 \cap \text{Der}(\mathfrak{n})_1$ , then  $D(E) \subset E$ ,  $D(E) \subset W$  and hence  $D(E) = 0$ . Since  $E$  and  $a, b, x, u, y$  generate  $\mathfrak{n}$  as a Lie algebra, it follows that  $D = 0$ .

Now, we will see that  $\text{Der}(\mathfrak{n})_0 + \text{Der}(\mathfrak{n})_1 = \text{Der}(\mathfrak{n})$ . Given  $D \in \text{Der}(\mathfrak{n})$ , let  $A = p_E \circ D|_E \in \mathfrak{gl}(E)$  where  $p_E$  the projection over  $E$  with respect to the decomposition  $\mathfrak{n} = E \oplus W$ . Let  $D_0 \in \text{Der}(\mathfrak{n})_0$  be the derivation associated to  $A$ . Since the matrix of  $D_0$  in a basis  $\mathfrak{B}$  is of the form (3.1), it follows that  $D_1 = D - D_0 \in \text{Der}(\mathfrak{n})_1$ . This proves (c).  $\square$

We are now in a position to prove the main result of this subsection.

**Theorem 3.8.** *Let  $D \in \text{Der}(\mathfrak{n})$  be a diagonalizable derivation with eigenvalues 1, 2 and 3, then the dimension of the eigenspaces are  $d_1$ ,  $d_2$  and  $d_3$  respectively (see Proposition 3.2). In particular, if  $\tilde{\mathfrak{n}}_1 \oplus \tilde{\mathfrak{n}}_2 \oplus \tilde{\mathfrak{n}}_3$  is any grading of  $\mathfrak{n}(n)$ , then  $\dim \tilde{\mathfrak{n}}_i = d_i$ ,  $i = 1, 2, 3$ .*

*Proof.* Suppose  $D = D_A + D_1$  where  $D_A \in \text{Der}(\mathfrak{n})_0$  and  $D_1 \in \text{Der}(\mathfrak{n})_1$ . Since  $D$  is diagonalizable, then  $A$  is diagonalizable as well. Then we can choose a basis  $B$  of  $E$  such that the matrix of  $D_A$  in the basis  $\mathfrak{B} = \mathfrak{B}(B)$  is diagonal (see (3.1)). Since  $D_1$  has a lower triangular matrix in the basis  $\mathfrak{B}$ , the matrix of  $D$  in this basis is lower triangular.

As in Proposition 3.6, for each  $v \in \mathfrak{B}$ , we denote by  $\lambda_v$  the diagonal coefficient of the matrix of  $D$  corresponding to the vector  $v$ . It is clear that  $\{\lambda_v : v \in \mathfrak{B}\}$  are the eigenvalues of  $D$  counted with multiplicity. Now,  $\lambda_v$  is either equal to 1, 2 or 3 for all  $v \in \mathfrak{B}$ . This, together with Proposition 3.6 and the shape of the matrix of  $D_A$ , implies that

$$\lambda_a = \lambda_b = \lambda_x = 1, \quad \lambda_y = \lambda_u = \lambda_c = 2, \quad \lambda_h = \lambda_f = 3.$$

Finally, since  $D$  is a Lie algebra homomorphism, we obtain that

$$\lambda_{u_i} = \lambda_{e_i} + \lambda_u, \quad \lambda_{y_i} = \lambda_{e_i} + \lambda_y, \quad \lambda_{x_i} = \lambda_{e_i} + \lambda_x, \quad \lambda_{e_i \wedge e_j} = \lambda_{e_i} + \lambda_{e_j}$$

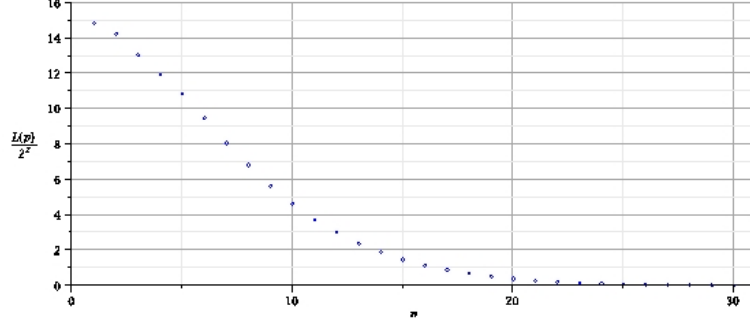
and hence  $\lambda_{e_i} = 1$ ,  $\lambda_{u_i} = 3$ ,  $\lambda_{y_i} = 3$ ,  $\lambda_{x_i} = 2$  and  $\lambda_{e_i \wedge e_j} = 2$ . Counting the number of eigenvalues, we obtain that the multiplicities of 1, 2 and 3 are respectively  $d_1$ ,  $d_2$  and  $d_3$ .  $\square$

**3.3. The Lie algebras  $\mathfrak{n}(n)$  are WSPT for  $n \geq 17$ .** As a consequence of Theorem 3.8, the numbers  $d_1$ ,  $d_2$  and  $d_3$  are independent of the grading of  $\mathfrak{n}(n)$ . Recall from Proposition 3.2 that

$$d_1 = n + 3; \quad d_2 = 3 + \frac{n(n+1)}{2}; \quad d_3 = 2(n+1),$$

$$d_2^0 = 2; \quad d_2^1 = 1; \quad z_2 = \frac{n(n+1)}{2}; \quad z = \frac{(n+4)(n+1)}{2}.$$

The following plot shows the quotient  $\frac{L(p)}{2^z}$ , and it can be proved (see below) that this quotient is a decreasing function of  $n$  that converges to 0 as  $n \rightarrow \infty$ .



In particular, we see that for  $n = 17$ ,  $\mathfrak{n}$  is WSPT. We have  $\dim \mathfrak{n} = 212$  and

$$d_1 = 20, \quad d_2 = 156, \quad d_3 = 36; \quad d_2^0 = 2, \quad d_2^1 = 1, \quad z_2 = 153; \quad z = 189.$$

**Theorem 3.9.** *If  $n \geq 17$ , the Lie algebra  $\mathfrak{n}(n)$  is WSPT.*

*Proof.* We need to prove that  $L\left(\frac{(1-x)^{n+3}(1-x^2)^{\frac{n(n+1)}{2}+3}(1-x^3)^{2n+2}}{2^{\frac{(n+4)(n+1)}{2}}}\right) < 1$  for  $n \geq 17$  (we checked it computationally for  $17 \leq n \leq 200$ ). We start by rearranging the factors of the polynomial in the following way:

$$\frac{(1-x)^n(1-x^2)^{2n}(1-x^3)^{2n}}{2^{4n}} \cdot \frac{(1-x^2)^{\frac{n(n-3)}{2}+3}}{2^{\frac{n(n-3)}{2}+3}} \cdot 2(1-x)^3(1-x^3)^2.$$

Since  $L\left(\frac{(1-x^2)^{\frac{n(n-3)}{2}+3}}{2^{\frac{n(n-3)}{2}+3}}\right) \leq 1$ ,  $L(2(1-x)^3(1-x^3)^2) \leq 64$ , and  $L(pq) \leq L(p)L(q)$ , we only need to show that

$$L(p_n) < \frac{1}{64}, \quad \text{where} \quad p_n(x) = \frac{(1-x)^n(1-x^2)^{2n}(1-x^3)^{2n}}{2^{4n}},$$

for  $n \geq 200$ . We checked computationally that  $L(p_n) < \frac{1}{2}$  for  $n = 30, \dots, 180$ . Now, if  $n > 180$ , then  $n - 150 > 30$ , and arguing by induction, we obtain

$$L(p_n) \leq L(p_{30})^5 L(p_{n-150}) \leq \left(\frac{1}{2}\right)^6 = \frac{1}{64}. \quad \square$$

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